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THE BRAIDING FOR REPRESENTATIONS OF q-DEFORMED AFFINE sl_2.

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ABSTRACT. We compute the braiding for the 'principal gradation' of $U_q(\widehat{sl_2})$ for $|q| = 1$ from first principles, starting from the idea of a rigid braided tensor category. It is not necessary to assume either the crossing or the unitarity condition, from S-matrix theory. We demonstrate the uniqueness of the normalisation of the braiding under certain analyticity assumptions, and show that its convergence is critically dependent on the number theoretic properties of the number τ in the deformation parameter $q = e^{2\pi i \tau}$. We also examine the convergence using probability, assuming a uniform distribution for q on the unit circle.

1 Introduction

The sine-Gordon S-matrix for the quantum scattering of solitons was solved initially in the literature in [9], by imposing conditions on the S-matrix called crossing and unitarity. These conditions were previously observed in other S-matrices found using perturbative techniques in quantum field theory [8], and were subsequently taken to be axioms for the non-perturbative result.

The bootstrap methods use the $U_q(\widehat{sl_2})$ Hopf algebra [3], and its spin half representation W, which is a space of functions taking values in $\mathbb{C}^2$, where the first component corresponds to 'solitons' and the second to 'anti-solitons'. Then $W \otimes W$ corresponds to a two soliton system, which can interact by collision. There is an initial two soliton quantum state in $W \otimes W$, and after the collision process we have a final quantum state in $W \otimes W$. The scattering matrix gives a map $W \otimes W \to W \otimes W$, which sends the initial to the final state. In terms of the Hopf algebra, this map is a braiding [7].

This scattering matrix is fairly easy to find up to a multiplication by a scalar function, but the scalar function itself is more difficult. We denote this scalar function by $a(z)$ where $z \in \mathbb{C}$. The crossing condition in terms of this function becomes

$$a(z) = a\left(-\frac{q}{z}\right) \frac{(z - z^{-1})}{(zq^{-1} - z^{-1}q)} \tag{1}$$

and the unitarity condition is $a(z)a(z^{-1}) = 1$.

The crossing condition arises from physics by equating a scattering process with the same scattering process after rotating the space-time diagram of the collision by a right-angle. This rotation involves a time reversal of one of the incoming and one of the outgoing solitons, which implies that these are turned into anti-solitons. The rotation of the diagram also implies that z, which corresponds roughly to the relative momentum of the colliding solitons (conserved in the collision), is transformed to $-z^{-1}q$. The interested reader can refer to the comprehensive book [8] for a complete discussion. The article [9] is also an excellent review.

Zamolodchikov-Zamolodchikov[9] solved these equations (1) to get a formula for $a(z)$ in terms of a double infinite product of gamma functions. It was thought that this product probably

converged for all physical values of z, and all values of q where $|q| = 1$. However no proof of this was provided. The convergence of this function has also not been treated in subsequent papers in the literature.

Here, in this paper, we find an alternative formula which is more amenable to a convergence analysis. We find that the convergence is highly delicate, and the function converges with probability one, for q on the unit circle, i.e. $q = e^{2\pi i \tau}$, including convergence for all irrational algebraic values of τ, and diverges for certain transcendental values of τ.

Another alternative formula for $a(z)$ was found in Johnson [6], as an integral, or as a combination of 'regularised' quantum dilogarithms. This was done to make contact with semi-classical results for the scattering which involve integrating classical time delays. However, the convergence of this formula, or of the individual quantum dilogarithms, was also difficult to analyse because the contour integrals which one has to do are difficult to perform, involving sums over an infinite double set of poles.

In this paper we shall consider the problem purely in terms of braidings of the representations of a Hopf algebra, rather than invoking the crossing and unitarity conditions of S-matrix theory. We begin with the loop group of analytic functions used in the classical inverse scattering procedure for sine-Gordon [1], and deform this by inclusion of a parameter q [2].

2 The universal enveloping algebra $\mathcal{H}$

We begin with the loop group of analytic functions from $\mathbb{C}^*$ to $SL_2(\mathbb{C})$ which obey the symmetry condition $U\phi(-z)U^{-1} = \phi(z)$, where

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The Lie algebra for this group has generators $X_{\pm 1}$, $X_{\pm 2}$ and H, given by

$$X_{+1}(z) = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \quad , \quad X_{-1}(z) = \begin{pmatrix} 0 & 1/z \\ 0 & 0 \end{pmatrix} \quad , \quad H(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$
$$X_{+2}(z) = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \quad , \quad X_{-2}(z) = \begin{pmatrix} 0 & 0 \\ 1/z & 0 \end{pmatrix} . \tag{2}$$

These generators obey the usual coproduct rule for the universal enveloping algebra of a Lie algebra, namely $\Delta(X_{\pm 1}) = X_{\pm 1} \otimes 1 + 1 \otimes X_{\pm 1}$ etc. This rule can be deformed by the inclusion of a parameter $q \in \mathbb{C}$ to give

$$\Delta H = 1 \otimes H + H \otimes 1 \quad , \quad \Delta X_{\pm 1} = X_{\pm 1} \otimes q^{-H/2} + q^{H/2} \otimes X_{\pm 1} ,$$
$$\Delta X_{\pm 2} = X_{\pm 2} \otimes q^{H/2} + q^{-H/2} \otimes X_{\pm 2} . \tag{3}$$

The Lie algebra structure remains the same. If in addition we define a counit ϵ (which kills all the generators) and an antipode S (which has $S(H) = -H$ and $S(X_{\pm n}) = -q^{\pm 1}X_{\pm n}$) we can make the universal enveloping algebra into a Hopf algebra, which we denote $\mathcal{H}$. This is the so-called 'principal gradation' of $U_q(\widehat{sl_2})$, which is actually a subalgebra of $U_q(\widehat{sl_2})$. We use the convention that $r = \sqrt{q}$ in the terms $q^{\pm H/2}$.

3 Rigid braided tensor categories

Here we shall give a highly abbreveiated, specialised and incomplete account of rigid braided tensor categories, for a full account see [7].

The representations of a Hopf algebra $\mathcal{H}$ form a category, with objects the representations, and the morphisms $\rho : V \to W$ are intertwining maps for the representations V and W. This means that ρ is linear and that $\rho(h(v)) = h(\rho(v))$ for all $h \in \mathcal{H}$.

The tensor product of two representations is also a representation. The action on $V \otimes W$ is given by the coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$. If we write $\Delta h = \sum h_{(1)} \otimes h_{(2)}$, then $h(v \otimes w) = \sum h_{(1)}(v) \otimes h_{(2)}(w)$. The category contains an 'identity object', the representation $\mathbb{C}$ with all generators having zero action. This means that category of representations forms a tensor or monoidal category. (Technically we should also say that the associator is trivial.)

If the category is *rigid*, for an object V there is a dual object V', and an 'evaluation' morphism eval $: V' \otimes V \to \mathbb{C}$ given by $\mathrm{eval}(\alpha, v) = \alpha(v)$.

If the category is *braided*, for two objects V and W there is a morphism $\Psi_{VW} : V \otimes W \to W \otimes V$. The braiding is functorial, which means that if there is a morphism $\theta : V \to X$, then the maps $(I \otimes \theta)\Psi_{VW} : V \otimes W \to W \otimes X$ and $\Psi_{XW}(\theta \otimes I) : V \otimes W \to W \otimes X$ are the same.

Figure 1 shows the standard diagramatic notation used for braided categories. Elements of representations are denoted by vertical lines. For evaluation (a), note that $\mathbb{C}$ is traditionally denoted by an invisible line. The braiding $\Psi_{VW} : V \otimes W \to W \otimes V$ is shown in (b), and the rule for the functoriality of the braiding in (c).
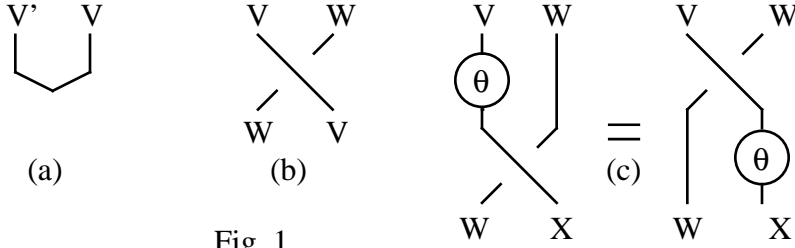

Fig. 1.

The finite dimensional representations of a quasitriangular Hopf algebra form a rigid braided tensor category. We shall assume that the representations of our Hopf algebra $\mathcal{H}$ also form a rigid braided tensor category, and this will allow us to explicitly calculate the braiding.

4 The 'standard' representation of $\mathcal{H}$

Take W to be a vector space of analytic functions $: \mathbb{C}^* \to \mathbb{C}^2$ (or at least analytic in a neighbourhood of zero and a neighbourhood of infinity), which obeys the condition $Uw(-z) = w(z)$ for all $w \in W$. The algebra $\mathcal{H}$ acts on W using matrix multiplication, $(hw)(z) = h(z)w(z)$, for the five generators listed in (2).

We can consider a dual space W' to W, which shall consist of analytic functions $: \mathbb{C}^* \to \mathbb{C}^2$ which are defined for $|z|$ sufficiently small and sufficiently large. Now define an evaluation map eval $: W' \otimes W \to \mathbb{C}$ by

$$\mathrm{eval}\left(\begin{pmatrix} f \\ g \end{pmatrix} \otimes \begin{pmatrix} u \\ v \end{pmatrix} \right) = \frac{1}{4\pi i} \oint_\gamma \left(f(z)\, u(z) + g(z)\, v(z) \right) \frac{dz}{z}, \tag{4}$$

where γ consists of two anticlockwise circular contours about 0, one of large radius, and one of small radius. To find the action of $\mathcal{H}$ on W' we use the fact that the action commutes with eval $: W' \otimes W \to \mathbb{C}$, and that the action of the generators is zero on $\mathbb{C}$. This means that

$$\begin{aligned} 0 &= H\Big(\mathrm{eval}\Big(\begin{pmatrix} f \\ g \end{pmatrix} \otimes \begin{pmatrix} u \\ v \end{pmatrix} \Big)\Big) \\ &= \mathrm{eval}\Big(H\Big(\begin{pmatrix} f \\ g \end{pmatrix} \otimes \begin{pmatrix} u \\ v \end{pmatrix} \Big)\Big) = \mathrm{eval}\Big(H \begin{pmatrix} f \\ g \end{pmatrix} \otimes \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix} \otimes H \begin{pmatrix} u \\ v \end{pmatrix} \Big), \end{aligned}$$

so we get

$$\mathrm{eval}\Big(H \begin{pmatrix} f \\ g \end{pmatrix} \otimes \begin{pmatrix} u \\ v \end{pmatrix} \Big) = -\,\mathrm{eval}\Big(\begin{pmatrix} f \\ g \end{pmatrix} \otimes \begin{pmatrix} u \\ -v \end{pmatrix} \Big)$$

$$= -\frac{1}{4\pi i} \oint \Big(f(z)\, u(z) - g(z)\, v(z) \Big) \frac{dz}{z}\,,$$

so we deduce that

$$H \begin{pmatrix} f \\ g \end{pmatrix}(z) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}.$$

Now we continue with the generator X_{+1},

$$\begin{aligned}
0 &= \mathrm{eval}\Big(X_{+1}\Big(\begin{pmatrix} f \\ g \end{pmatrix} \otimes \begin{pmatrix} u \\ v \end{pmatrix}\Big)\Big) \\
&= \mathrm{eval}\Big(X_{+1} \begin{pmatrix} f \\ g \end{pmatrix} \otimes \begin{pmatrix} u/r \\ vr \end{pmatrix} + \begin{pmatrix} f/r \\ gr \end{pmatrix} \otimes X_{+1} \begin{pmatrix} u \\ v \end{pmatrix}\Big)\,,
\end{aligned}$$

from which we get

$$\mathrm{eval}\Big(X_{+1} \begin{pmatrix} f \\ g \end{pmatrix} \otimes \begin{pmatrix} u/r \\ vr \end{pmatrix}\Big) = -\mathrm{eval}\Big(\begin{pmatrix} f/r \\ gr \end{pmatrix} \otimes \begin{pmatrix} 0 \\ zu \end{pmatrix}\Big) = -\frac{r}{4\pi i} \oint z\, g(z)\, u(z) \frac{dz}{z}\,.$$

From this, and the corresponding calculations for the other generators, we see that the action on W' is given by the matrix multiplication

$$\Big(h \begin{pmatrix} f \\ g \end{pmatrix}\Big)(z) = \tilde{h}(z) \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}, \tag{5}$$

where the matrices $\tilde{h}(z)$ are given by

$$\tilde{X}_{+1}(z) = -q \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}, \quad \tilde{X}_{-1}(z) = -q^{-1} \begin{pmatrix} 0 & 0 \\ 1/z & 0 \end{pmatrix}, \quad \tilde{H}(z) = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\tilde{X}_{+2}(z) = -q \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}, \quad \tilde{X}_{-2}(z) = -q^{-1} \begin{pmatrix} 0 & 1/z \\ 0 & 0 \end{pmatrix}. \tag{6}$$

There is a morphism $\theta : W \to W'$ given by

$$\theta(k)(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} k(-zq)\,. \tag{7}$$

To verify this we check that it commutes with the actions of the generators, for example;

$$\begin{aligned}
(X_{+1}\theta(\begin{pmatrix} u \\ v \end{pmatrix}))(z) &= -\begin{pmatrix} 0 & qz \\ 0 & 0 \end{pmatrix} \theta(\begin{pmatrix} u \\ v \end{pmatrix})(z) = -\begin{pmatrix} 0 & qz \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v(-zq) \\ u(-zq) \end{pmatrix}, \\
\theta(X_{+1} \begin{pmatrix} u \\ v \end{pmatrix})(z) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (X_{+1} \begin{pmatrix} u \\ v \end{pmatrix})(-zq) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -qz & 0 \end{pmatrix} \begin{pmatrix} u(-zq) \\ v(-zq) \end{pmatrix},
\end{aligned}$$

which are equal as required.

5 The braidings

We use the convention for tensor products that $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$ and $M_2 \otimes M_2 \cong M_4$, where

$$\begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} u' \\ v' \end{pmatrix} \cong \begin{pmatrix} uu' \\ uv' \\ vu' \\ vv' \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \cong \begin{pmatrix} aa' & ab' & ba' & bb' \\ ac' & ad' & bc' & bd' \\ ca' & cb' & da' & db' \\ cc' & cd' & dc' & dd' \end{pmatrix}. \tag{8}$$

Then we can consider $W \otimes W$ or $W' \otimes W$ as a space of analytic maps from a subset of $\mathbb{C}^* \times \mathbb{C}^*$ to $\mathbb{C}^4$. Now if $k \in W' \otimes W$ we have

$$\mathrm{eval}(k) = \frac{1}{4\pi i} \oint (1 \quad 0 \quad 0 \quad 1)\, k(x, x) \frac{dx}{x}\,.$$

4

By the coproduct rule H and (for example) X_{+1} act on $W \otimes W$ by matrix multiplication

$$
\begin{aligned}
(Hk)(x,y) &= (H(x) \otimes I_2 + I_2 \otimes H(y))k(x,y) \ , \\
(X_{+1}k)(x,y) &= (X_{+1}(x) \otimes q^{-H(y)/2} + q^{H(x)/2} \otimes X_{+1}(y))\, k(x,y) \ ,
\end{aligned}
$$

and they act on $W' \otimes W$ by

$$
\begin{aligned}
(Hk)(x,y) &= (\tilde{H}(x) \otimes I_2 + I_2 \otimes H(y))k(x,y) \ , \\
(X_{+1}k)(x,y) &= (\tilde{X}_{+1}(x) \otimes q^{-H(y)/2} + q^{\tilde{H}(x)/2} \otimes X_{+1}(y))\, k(x,y) \ .
\end{aligned}
$$

The braiding $\Psi_{WW} : W \otimes W \to W \otimes W$ will be assumed to have the form $(\Psi_{WW}k)(x,y) = M(x,y)k(y,x)$, where $M(x,y)$ is a 4×4 matrix. Since the braiding is a morphism we must have $\Psi_{WW}(hk) = h(\Psi_{WW}k)$ for the five generators h and all $k \in W \otimes W$. The cases for $h = H$ and X_{+1} are given below:

$$
\begin{aligned}
M(x,y)\,(H(y) \otimes I_2 + I_2 \otimes H(x)) &= (H(x) \otimes I_2 + I_2 \otimes H(y))\, M(x,y) \ , \\
M(x,y)\,(X_{+1}(y) \otimes q^{-H(x)/2} + q^{H(y)/2} \otimes X_{+1}(x)) &= (X_{+1}(x) \otimes q^{-H(y)/2} + q^{H(x)/2} \otimes X_{+1}(y))\, M(x,y).
\end{aligned}
$$

A simple calculation will show that these five conditions determine the matrix $M(x,y)$ up to a complex multiple, and we find

$$
(\Psi_{WW}k)(x,y) \;=\; a(x,y)
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{xy(q^2-1)}{q^2x^2-y^2} & \frac{q(x^2-y^2)}{q^2x^2-y^2} & 0 \\
0 & \frac{q(x^2-y^2)}{q^2x^2-y^2} & \frac{xy(q^2-1)}{q^2x^2-y^2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
k(y,x) \ ,
$$

where $a(x,y)$ is complex valued. Note that $(\Psi_{WW}^2 k)(x,y) = a(x,y)a(y,x)k(x,y)$. In the same manner we can determine the braiding $\Psi_{W'W} : W' \otimes W \to W \otimes W'$ to be $(\Psi_{W'W}k)(x,y) = N(x,y)k(y,x)$, where the matrix $N(x,y)$ is given by

$$
(\Psi_{W'W}k)(x,y) \;=\; \frac{c(x,y)}{q^2y^2-x^2}
\begin{pmatrix}
q^2y^2-x^2 & 0 & 0 & (q^2-1)xy \\
0 & 0 & q(y^2-x^2) & 0 \\
0 & q(y^2-x^2) & 0 & 0 \\
(q^2-1)xy & 0 & 0 & q^2y^2-x^2
\end{pmatrix}
k(y,x) \ ,
$$

where $c(x,y)$ is another complex valued function.

Now we use the fact that as the braiding is functorial, it must commute with the evaluation morphism. We see that the maps $(I \otimes \mathrm{eval})(\Psi_{W'W} \otimes I)(I \otimes \Psi_{WW})$ and $\mathrm{eval} \otimes I : W' \otimes W \otimes W \to W$ are the same. In terms of the standard pictures, this is fig. 2.
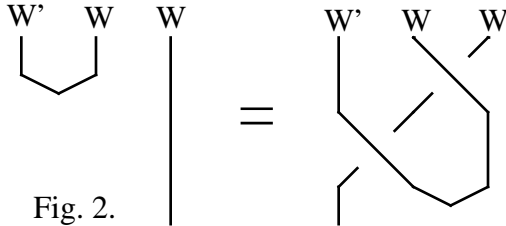

Fig. 2.

Now, identifying $W' \otimes W \otimes W$ with maps from subsets of $(\mathbb{C}^*)^3$ to $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, we get

$$
\begin{aligned}
((\Psi_{W'W} \otimes I)(I \otimes \Psi_{WW})k)(x,y,z) &= (N(x,y) \otimes I_2)((I \otimes \Psi_{WW})k)(y,x,z) \\
&= (N(x,y) \otimes I_2)\,(I_2 \otimes M(x,z))\,k(y,z,x) \ , \qquad (9)
\end{aligned}
$$

and applying $I \otimes \mathrm{eval}$ to this gives

$$\frac{1}{4\pi i} \oint (I_2 \otimes (\begin{matrix} 1 & 0 & 0 & 1 \end{matrix}))(N(x,z) \otimes I_2)(I_2 \otimes M(x,z)) \, k(z,z,x) \, \frac{dz}{z} \ ,$$

and some matrix multiplication shows that this is

$$\frac{1}{4\pi i} \oint a(x,z) \, c(x,z) \, ((\begin{matrix} 1 & 0 & 0 & 1 \end{matrix}) \otimes I_2) \, k(z,z,x) \, \frac{dz}{z} \ .$$

Just applying $\mathrm{eval} \otimes I$ to k gives

$$\frac{1}{4\pi i} \oint ((\begin{matrix} 1 & 0 & 0 & 1 \end{matrix}) \otimes I_2) \, k(z,z,x) \, \frac{dz}{z} \ ,$$

and since these must be the same for all choices of k we deduce that $c(x,z) = 1/a(x,z)$.

Now use the fact that the braiding commutes with the morphism $\theta : W \to W'$ in (7), so $(I \otimes \theta)\Psi_{WW}$ and $\Psi_{W'W}(\theta \otimes I) : W \otimes W \to W \otimes W'$ are the same. Then

$$\begin{aligned}
((I \otimes \theta)\Psi_{WW}k)(x,y) &= (I_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})(\Psi_{WW}k)(x,-qy) \\
&= (I_2 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \, M(x,-qy) \, k(-qy,x) \ , \\
(\Psi_{W'W}(\theta \otimes I)k)(x,y) &= N(x,y) \, ((\theta \otimes I)k)(y,x) \ , \\
&= N(x,y) \, (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_2) \, k(-qy,x) \ .
\end{aligned} \tag{10}$$

From some more matrix multiplication, this is true if

$$\frac{1}{a(x,y)} = a(x,-qy) \, \frac{x^2 - q^2 y^2}{q(x^2 - y^2)} \ . \tag{11}$$

6   The normalisation of the braiding, $|q| = 1$.

In this section we find the normalisation $a(x,y)$. For the moment we shall suppose that the value of x is fixed. Suppose that a solution $a(x,y)$ of (11) is an analytic function of y (except for isolated singularities) in some annulus centered on zero contained in the region $|y| > |x|$. We can narrow the annulus down until it no longer contains any isolated singularities or zeros. For convenience we set $z = x/y$ (so $|z| < 1$), and $f(z) = a(x,x/z)$. Then $f(z)$ satisfies the equation

$$f(z)f(-z/q) = q\frac{z^2 - 1}{z^2 - q^2} \ . \tag{12}$$

Now $f(z)$ will have a winding number $\omega \in \mathbb{Z}$ around zero as z winds once around zero. The function $c(z) = z^{-\omega}f(z)$ will have zero winding number, so its log, $b(z) = \log(c(z))$, will be analytic (and single valued) in the annulus. Now $c(z)$ obeys the equation $c(z)c(-z/q) = q(-z^2/q)^{-\omega}(z^2 - 1)/(z^2 - q^2)$, so we must have

$$b(z) + b(-z/q) = \log(1 - z^2) - \log(1 - z^2/q^2) - \log(q) - \omega \log(-z^2/q) \ ,$$

as $-z/q$ is in the annulus if z is, because $|q| = 1$. Since all the other functions are single valued in the annulus, we must have $\omega = 0$. Now we can take the Laurent expansion of both sides in the annulus, and compare coefficients, to get the *unique* solution (up to the addition of a multiple of πi)

$$b(z) = -\frac{1}{2}\log(q) + \sum_{n>0} \frac{1}{n}\frac{1 - q^{2n}}{1 + q^{2n}} z^{2n} \ , \quad |z| < 1 \ , \ z \in \text{annulus} \ . \tag{13}$$

6

By the uniqueness of analytic continuation, the exponential of this formula must coincide with $f(z)$ in a disk from zero up to the radius of convergence of the series. We also conclude that $f(z)$ did in fact not have any zeros or isolated singularities in this disk.

In the same manner, if $f(z)$ were analytic (except for isolated singularities) in some annulus centered on zero outside the unit disk, we could conclude that on that annulus (again up to the addition of a multiple of πi)

$$b(z) = \frac{1}{2}\log(q) - \sum_{n>0} \frac{1}{n}\frac{1-q^{2n}}{1+q^{2n}} z^{-2n}, \quad |z| > 1. \tag{14}$$

We can now see explicitly from the series (on the assumption that the series converge anywhere) that $b(z) = -b(1/z)$ plus a multiple of πi, i.e. that $1/f(z) = \pm f(1/z)$.

Now we can look at different values of x. As a consequence of the analyticity condition we assumed for $a(x,y)$, it can be seen that $a(x,y)$ in fact only depends on $z = x/y$. We shall abuse our previous notation by referring to $a(z)$.

7 The convergence of the series by number theory, $|q| = 1$.

The radius of convergence R of the series (13) is given by

$$R^{-1} = \limsup_{n\to+\infty} \left| \frac{1}{n} \cdot \frac{1-q^{2n}}{1+q^{2n}} \right|^{1/(2n)} = \limsup_{n\to+\infty} \left| \frac{1-q^{2n}}{1+q^{2n}} \right|^{1/(2n)}. \tag{15}$$

We see that the series is not even defined if q is an even root of -1. For any other root of unity the series has radius of convergence 1, except for $q = \pm 1$, when the series terminates. Now consider the case $q = e^{2\pi i\tau}$, where τ is irrational.

Call $n \in \mathbb{N}$ type 1 if $|1 - q^{2n}| < 1$, which implies that $|1 + q^{2n}| \geq 1$. Then

$$\limsup_{n\to+\infty,\,\text{type 1}} \left| \frac{1-q^{2n}}{1+q^{2n}} \right|^{1/(2n)} \leq \limsup_{n\to+\infty,\,\text{type 1}} 1 = 1.$$

Call $n \in \mathbb{N}$ type 2 if $|1 - q^{2n}| \geq 1$, in which case

$$\limsup_{n\to+\infty,\,\text{type 2}} \left| \frac{1-q^{2n}}{1+q^{2n}} \right|^{1/(2n)} \geq \limsup_{n\to+\infty,\,\text{type 2}} \left| \frac{1}{1+q^{2n}} \right|^{1/(2n)} \geq 1,$$

and we deduce that

$$R^{-1} = \limsup_{n\to+\infty,\,\text{type 2}} \left| \frac{1-q^{2n}}{1+q^{2n}} \right|^{1/(2n)}.$$

Then we find

$$\limsup_{n\to+\infty,\,\text{type 2}} \left| \frac{1}{1+q^{2n}} \right|^{1/(2n)} \leq R^{-1} \leq \limsup_{n\to+\infty,\,\text{type 2}} \left| \frac{2}{1+q^{2n}} \right|^{1/(2n)},$$

which implies

$$R^{-1} = \limsup_{n\to+\infty,\,\text{type 2}} \left| \frac{1}{1+q^{2n}} \right|^{1/(2n)} = \limsup_{n\to+\infty} \left| \frac{1}{1+q^{2n}} \right|^{1/(2n)}. \tag{16}$$

If we let $d(\tau, n)$ be the minimum distance from $4\pi in\tau$ to an odd multiple of πi, then $d(\tau,n) \geq |1+q^{2n}| \geq \frac{2}{\pi}d(\tau,n)$, so

$$R = \liminf_{n\to+\infty} \left(\min_{p\,\text{odd}} |4\pi in\tau - p\pi i| \right)^{1/(2n)} = \liminf_{n\to+\infty} \left(\min_{p\,\text{odd}} \left| \tau - \frac{p}{4n} \right| \right)^{1/(2n)} \leq 1. \tag{17}$$

We see that the radius of convergence of the power series is dependent on how well τ can be approximated by rational numbers. Fortunately many results are known in this area [5], and we shall use one of these now.

Suppose that the irrational number $\tau \in \mathbb{R}$ is *algebraic* of degree $k > 1$ (this means that it is a root of a polynomial of degree k with integer coefficients). Then there is a constant $K > 0$ so that for all n and all $p \in \mathbb{Z}$,

$$|\tau - \frac{p}{4n}| \geq \frac{K}{(4n)^k} .$$

From the formula for R above (17),

$$R \geq \liminf_{n \to +\infty} \left(\frac{K}{2^k \, (2n)^k} \right)^{1/(2n)} = 1 ,$$

so we conclude that for any irrational algebraic number τ, the radius of convergence is 1.

To get a radius of convergence less than 1, we shall have to create an irrational number with very good rational approximations. Let

$$\tau = \sum_{s \geq 1} \frac{1}{4m_s} , \tag{18}$$

where the strictly positive integers m_s have the property that $4m_s$ divides m_{s+1} for all $s \geq 1$. If we set $n = m_t$, then

$$\frac{n}{m_{t+1}} \leq |l - 4n\tau| \leq \frac{2n}{m_{t+1}}$$

for $l = \sum_{t \geq s \geq 1} m_t/m_s$ an odd integer, so

$$\frac{1}{4m_{t+1}} \leq \min_{p \text{ odd}} |\tau - \frac{p}{4n}| \leq \frac{1}{2m_{t+1}} .$$

Then by (17),

$$R \leq \liminf_{t \to \infty} \left(\frac{1}{m_{t+1}} \right)^{1/(2m_t)} .$$

If we set $m_1 = 1$ and $m_{s+1} = 2^{2sm_s}$ for all $s \geq 1$, then the radius of convergence of the series (13) for τ given by (18) is zero. Also τ is too closely approximated by non-equal rational numbers to be rational itself.

8 The convergence of the series by probability, $|q| = 1$.

We consider the probability that the radius of convergence R of the series (13) is 1, given that q has a uniform distribution on the circle (equivalently, τ has a uniform distribution on $[0, 1]$). From (17), for $0 < s < 1$:

$$P[R > s] = P[\liminf_{n \to \infty} \min_{p \text{ odd}} |\tau - \frac{p}{4n}|^{1/(2n)} > s] = \lim_{m \to \infty} P[\inf_{n \geq m} \min_{p \text{ odd}} |\tau - \frac{p}{4n}|^{1/(2n)} > s] . \tag{19}$$

For any random variable X_n and $s < t < 1$,

$$(\forall n \geq m \quad X_n > t) \Rightarrow \inf_{n \geq m} X_n > s ,$$

or alternatively

$$P[X_n > t \quad \forall n \geq m] \leq P[\inf_{n \geq m} X_n > s] .$$

Then from (19),

$$P[R > s] \geq \lim_{m \to \infty} P[\min_{p \text{ odd}} |\tau - \frac{p}{4n}|^{1/(2n)} > t \quad \forall n \geq m] .$$

If we define

$$A_n(x) = \{\tau \in [0,1] : \min_{p \text{ odd}} |\tau - \frac{p}{4n}| \leq x\} ,$$

then (with superscript c denoting complement)

$$P[R > s] \geq \lim_{m \to \infty} P[\cap_{n \geq m} A_n(t^{2n})^c] .$$

Then by taking complements

$$P[R \leq s] \leq \lim_{m \to \infty} P[\cup_{n \geq m} A_n(t^{2n})] \leq \lim_{m \to \infty} \sum_{n \geq m} P[A_n(t^{2n})] ,$$

so if the series

$$\sum_{n \geq 1} P[A_n(t^{2n})] \tag{20}$$

converges, then $R > s$ with probability one. If we write

$$A_n(x) = \{\tau \in [0,1] : \min_{p \text{ odd}} |4n\tau - p| \leq 4nx\} ,$$

then, if $4nx \leq 1$, in the interval $[0, 4n]$ there are $2n$ odd integers p, and each has a disjoint interval of length $8nx$ about it satisfying the inequality above. From this we find $P[A_n(x)] = 4nx$, so the sum (20) becomes (with the exception of a finite number of terms at the beginning),

$$\sum_{n \geq 1} 4n \, t^{2n} ,$$

which converges since $|t| < 1$. We conclude that $P[R > s] = 1$, and so $R = 1$ with probability one. Since the algebraic numbers have measure zero, the series must have $R = 1$ for many transcendental (non-algebraic) τ.

9   The case $|q| \neq 1$.

In this case we would get the same unique series *if* the annulus in which $a(z)$ was analytic and free of zeros or singularities was sufficiently wide. We would need to have both z and $-z/q$ in the same annulus, so the ratio of the outer and inner radii of the annulus would have to be greater than the larger of $|q|$ and $1/|q|$. If this condition was satisfied, we would have the solutions (13) or (14). However there may be other solutions to the normalisation which would always have zeros or singularities in such wide annuli.

Let us examine the series (13) for $|q| \neq 1$. Here $\lim_{n \to +\infty} |(1 - q^{2n})/(1 + q^{2n})| = 1$, so the series has radius of convergence 1. However now we can make an analytic continuation of the series. In the case $|q| < 1$ we write, for any integer $k \geq 1$,

$$\frac{1 - q^{2n}}{1 + q^{2n}} = 1 + 2 \sum_{m=1}^{k} (-1)^m q^{2mn} - \frac{2 \, (-1)^k \, q^{2(k+1)n}}{1 + q^{2n}} , \tag{21}$$

and substituting this into (13) gives

$$b(z) = -\frac{1}{2} \log(q) - \log(1 - z^2) - 2 \sum_{m=1}^{k} (-1)^m \log(1 - z^2 q^{2m}) - \sum_{n \geq 1} \frac{2 \, (-1)^k \, q^{2(k+1)n} \, z^{2n}}{n \, (1 + q^{2n})} .$$

9

The last term in the formula tends to zero uniformly on any bounded set as $k \to \infty$, allowing us to take the limit as $k \to \infty$ of the other terms. We then get the infinite product expansion, valid everywhere in $\mathbb{C}^*$,

$$a_+(z) \;=\; \frac{1}{r} \cdot \frac{1}{(1-z^2)} \cdot \prod_{\text{odd } m \geq 1} \frac{(1-z^2 q^{2m})^2}{(1-z^2 q^{2(m+1)})^2} \;\;,\;\; |q| < 1 \;. \tag{22}$$

A similar rearrangement to (21) for $|q| > 1$ would give

$$a_+(z) \;=\; \frac{(1-z^2)}{r} \cdot \prod_{\text{odd } m \geq 1} \frac{(1-z^2 q^{-2(m+1)})^2}{(1-z^2 q^{-2m})^2} \;\;,\;\; |q| > 1 \;. \tag{23}$$

These are the *unique* (up to a sign) solutions of (12) which are analytic for z in a neighbourhood of zero in $\mathbb{C}^*$. In the same way we can analytically extend the series (14) to obtain $a_-(z) = \pm 1/a_+(1/z)$, the *unique* solutions of (11) which are analytic for z in a neighbourhood of infinity in $\mathbb{C}^*$.

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